

Rough Isometries of Lipschitz Function Spaces

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Abstract

We show that rough isometries between metric spaces X, Y can be lifted to the spaces of real valued 1-Lipschitz functions over X and Y with supremum metric and apply this to their scaling limits. For the inverse, we show how rough isometries between X and Y can be reconstructed from structurally enriched rough isometries between their Lipschitz function spaces.

1 Introduction

Is there a qualitative difference between functions on a continuum X and functions on a discrete set Y , which is ϵ -dense in X ? Obviously, singularities may emerge on X . But, apart from that, are there more differences? In this paper we want to show that, when we are concerned with 1-Lipschitz functions, nothing new is added in the passage from discrete to continuum. Even more, the coarse geometry of a metric space (in the sense of its rough isometries) is already determined by its 1-Lipschitz function space.

There are detailed books and lots of articles on many aspects of Lipschitz function spaces, including isometries between them. A nice survey is Weaver's book on *Lipschitz Algebras* [W] (cf. section 2.6, where Weaver elaborates on the exact same questions we want to tackle here, but in a non-coarse context and under different conditions). However, to the knowledge of the author, no book nor paper dealt with their coarse geometry yet. On the other hand, Lipschitz functions naturally appear in many aspects of coarse geometry, like the Levy concentration phenomenon or the definition of Lipschitz-Hausdorff distance in [G2]. But they are not dealt with as metric spaces either.

The two main theorems we want to show are as following:

Theorem 1 *Let X, Y be (possibly infinite) metric spaces, $\epsilon \geq 0$. For each ϵ -isometry $\eta : X \rightarrow Y$, there is a 4ϵ -ml-isomorphism $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ such that κ is ϵ -near $f \mapsto f \circ \eta$ for all $f \in \text{Lip } Y$.*

Theorem 2 *Let X, Y be complete (possibly infinite) metric spaces and $\epsilon \geq 0$. For each ϵ -ml-isomorphism $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ there is a 88ϵ -isometry $\eta : X \rightarrow Y$, such that κ is 62ϵ -near $f \mapsto f \circ \eta$ for all $f \in \text{Lip } Y$.*

The theorems above can be seen as variants on two 60 year old theorems. The first is given by Hyers and Ulam in [HU]:

Theorem 3 (Hyers-Ulam) *Let K, K' be compact metric spaces and let E and E' be the spaces of all real valued continuous functions on K and K' , respectively. If $T(f)$ is a homomorphism of E onto E' which is also an ϵ -isometry, then there exists an isometric transformation $U(f)$ of E onto E' such that $\|U(f) - T(f)\| \leq 21\epsilon$ for all f in E . Corollary: The underlying metric spaces K and K' are homeomorphic.*

As we make use of the order-lattice structure of the Lipschitz function spaces in our proofs, the following theorem by Kaplansky [Kp] is related to our results as well, we state it in its formulation by Birkhoff ([B], 2nd ed. p. 175f.):

Theorem 4 (Kaplansky) *Any compact Hausdorff space is (up to homeomorphism) determined by the lattice of its continuous functions.*

The structure of this paper is as follows: In the first section we want to recall the notions of rough isometry, lattices, and Lipschitz functions, and our notation for metrics with infinity. In the second section we will proof some simple properties of Lipschitz function spaces and define what we call “ Λ -functions”. In the third and fourth section, we will proof the two main theorems of this paper. The fifth section presents an application in the context of scaling limits, giving a partial answer to the introductory question. The final section provides some conclusions and perspectives for future development of this subject.

1.1 Notation

Throughout this paper, let $Z := \mathbb{R}_{\geq 0} \cup \{\infty\}$ and X, Y be (possibly infinite) metric spaces in the following sense:

Definition 5 *A (possibly infinite) metric space (X, d) is a non-empty set X together with a mapping $d : X \times X \rightarrow Z$ which is positive-definite and fulfills the triangle-inequality. For this, we set $\infty + z = z + \infty = \infty$ and $z \leq \infty$ for all $z \in Z$ and call $z \in Z$ positive iff $z \neq 0$. (X, d) is called true metric space iff $d(x, x') \neq \infty$ for all $x, x' \in X$.*

We will make heavy use of the symbol

$$|d(x, x') - d(y, y')| \leq z$$

for some x, x', y, y' in X or Y and $z \in Z$. To make sense of this in case one of the distances becomes infinite, we define the former symbol to be equivalent to

$$d(x, x') \leq d(y, y') + z \quad \text{and} \quad d(y, y') \leq d(x, x') + z.$$

In particular, we find $|\infty - \infty| = 0$. This might seem unfamiliar. Note however, that $|z - z'|$ can be perfectly understood as a (possibly infinite) metric on Z itself. (Note that there is no non-trivial convergence to ∞ in this metric, ∞ is just an infinitely far away point.)

In most cases we call metrics on X and Y both “ d ” as it should be clear from the elements which metric is meant.

Furthermore, it's obvious that a (possibly infinite) metric space X always is a disjoint union of true metric spaces X_j with $d(x, x') = \infty$ iff $x \in X_j, x' \in X_k$ with $j \neq k, j, k \in J$. We call the X_j *components* of X . We call X *complete*, iff all of its components are complete as true metric spaces.

Definition 6 Two (set theoretic) mappings $\alpha, \beta : X \rightarrow Y$ are ϵ -near to each other, $\epsilon \geq 0$, iff $d_Y(\alpha x, \beta x) \leq \epsilon \forall x \in X$. (We drop brackets where feasible.)

A (set theoretic) mapping $\alpha : X \rightarrow Y$ is ϵ -surjective, $\epsilon \geq 0$, iff for each $y \in Y$ there is $x \in X$ s.t. $d_Y(\alpha x, y) \leq \epsilon$.

Definition 7 A (not necessarily continuous) map $\eta : X \rightarrow Y$ is called an ϵ -isometric embedding, $\epsilon \geq 0$ (which shall always imply $\epsilon \in \mathbb{R}$), iff

$$|d_X(x, x') - d_Y(\eta x, \eta x')| \leq \epsilon$$

for all $x, x' \in X$.

A pair $\eta : X \rightarrow Y, \eta' : Y \rightarrow X$ of ϵ -isometric embeddings is called an ϵ -isometry (or rough isometry) iff $\eta \circ \eta'$ and $\eta' \circ \eta$ are ϵ -near the identities on Y and X , respectively. When we speak of an “ ϵ -isometry $\eta : X \rightarrow Y$ ” a corresponding map η' shall always be implied.

X and Y are called ϵ -isometric, iff there is an ϵ -isometry between them.

It's difficult to attribute the concept of rough isometry to a single person, as it was always present in the notion of quasi-isometry, which itself was an obvious generalization of what was then called pseudo-isometry by Mostow in his 1973-paper about rigidity (see [M], [G1], [Kn]). Recent developments about the stability of rough isometries can be found in [Ra].

Definition 8 A mapping $f : X \rightarrow Y$ is called (K, ϵ) -Lipschitz (i.e. “ K -Lipschitz map on ϵ -scale” in [G3]), $\epsilon, K \geq 0$, iff

$$d_Y(f(x), f(y)) \leq K \cdot d_X(x, y) + \epsilon \quad \forall x, y \in X.$$

If $\epsilon = 0$, f is K -Lipschitz (continuous). Define $\text{Lip}_{K, \epsilon}(X, Y)$ to be the set of all (K, ϵ) -Lipschitz functions $X \rightarrow Y$, and $\text{Lip}_{K, \epsilon} X := \text{Lip}_{K, \epsilon}(X, Z)$, $\text{Lip } X := \text{Lip}_{1, 0}(X)$.

If nothing else is said, Z is the default target space for a Lipschitz function.

Assume f to be a (K, ϵ) -Lipschitz function on X and $f(x) = \infty$ for some $x \in X$. Then clearly $f(y) = \infty$ for all y in finite distance to x . Thus, if X is a true metric space, we have $\text{Lip } X = \text{Lip}(X, \mathbb{R}_{\geq 0}) \cup \{\infty\}$.

Definition 9 A lattice (L, \wedge, \vee) is a set L together with two mappings $\wedge, \vee : L \times L \rightarrow L$ which are commutative, associative and fulfill the absorption laws $f \wedge (f \vee g) = f \vee (f \wedge g) = f$ for all $f, g \in L$. A lattice L is called complete iff all infima and all suprema of all subsets of L exist in L .

Of particular interest is $L = \text{Lip } X$ with \wedge and \vee pointwise minimum and maximum respectively, and \bigwedge, \bigvee pointwise infimum and supremum. The following proposition is a special case of Lemma 6.3 in [H] and Proposition 1.5.5 in [W]. To keep this article self-contained, we nevertheless give a proof:

Proposition 10 Let X be a (possibly infinite) metric space. Then $\text{Lip } X$ is complete as a lattice.

Proof Let $f_j, j \in J$ be in $\text{Lip } X$. Obviously, Z is complete as a lattice, with $\bigwedge_\emptyset = \infty$ and $\bigvee_\emptyset = 0$. So we define pointwise

$$g(x) := \bigvee_{j \in J} f_j(x), \quad h(x) := \bigwedge_{j \in J} f_j(x)$$

and observe that g and $h : X \rightarrow Z$ are Lipschitz: Let $x, y \in X$ be arbitrary. Then holds

$$h(x) \leq f_j(x) \leq f_j(y) + d(x, y)$$

for all $j \in J$, and thus, by passing to the infimum:

$$h(x) \leq h(y) + d(x, y).$$

Same for g . □

Example 11 The space $C([0, 1], [0, 1])$ of real-valued continuous functions $[0, 1] \rightarrow [0, 1]$ is not a complete lattice with pointwise minimum and maximum: Choose $f_j(x) = 0 \vee (1 - j \cdot x)$, $j \in \mathbb{N}$, the infimum is not continuous. The same example shows that the space $\bigcup_{K \geq 0} \text{Lip}_{K,0} X$ of all Lipschitz-functions with arbitrary Lipschitz constant is no complete lattice.

On $\text{Lip } X$, we consider the (possibly infinite) supremum metric

$$d_\infty(f, g) := \bigvee_{x \in X} |f(x) - g(x)|.$$

Note that $(\text{Lip } X, \wedge, \vee, d_\infty)$ is no metric lattice in the sense of Birkhoff [B]: There is no valuation on $\text{Lip } X$ inducing d_∞ , and property (4) of Theorem 1, p. 230 (third edition) is explicitly violated even by bounded Lipschitz functions.

2 Fundamental properties

Proposition 12 *For f_j, g_j arbitrary set theoretic functions $X \rightarrow Z$, $j \in J$, J some arbitrary index set, holds:*

$$\begin{aligned} d_\infty \left(\bigwedge_{j \in J} f_j, \bigwedge_{j \in J} g_j \right) &\leq \bigvee_{j \in J} d_\infty(f_j, g_j) \\ d_\infty \left(\bigvee_{j \in J} f_j, \bigvee_{j \in J} g_j \right) &\leq \bigvee_{j \in J} d_\infty(f_j, g_j) \end{aligned}$$

Proof For $J = \emptyset$, both inequalities are trivial. Assume $J \neq \emptyset$. As $\bigvee_{j \in J}$ and $\bigvee_{x \in X}$ commute, it suffices to show

$$\begin{aligned} d_\infty \left(\bigwedge x_j, \bigwedge y_j \right) &\leq \bigvee d_\infty(x_j, y_j) \\ d_\infty \left(\bigvee x_j, \bigvee y_j \right) &\leq \bigvee d_\infty(x_j, y_j) \end{aligned}$$

for any $x_j, y_j \in Z$.

First we handle infinities. First inequality: Assume there is j with $x_j = y_j = \infty$. We can ignore all such j 's from J , unless all x_j and y_j are ∞ . In this case on both sides are zeros. Now assume $x_j = \infty \neq y_j$. Then ∞ appears on the right side and trivializes the inequality. So we can restrict to finite x_j and y_j . Note that $\bigwedge_j x_j = \infty$ can only happen when all $x_j = \infty$.

Second inequality: Assume $\bigvee_j x_j = \infty$, but $\bigvee_j y_j$ is finite. Then there is an upper bound for y_j but not for x_j . Hence the right side becomes infinite, too. Note that infinite x_j or y_j automatically lead to infinite $\bigvee_j x_j$ or $\bigvee_j y_j$, respectively.

Without restriction let $\bigwedge_j x_j \geq \bigwedge_j y_j$, and let $M := \bigvee_j d(x_j, y_j)$. Let $\delta > 0$ be arbitrary. Then there is an $m \in J$ with $y_m \leq \bigwedge_j y_j + \delta$. Furthermore we have $d(x_m, y_m) \leq M$, hence $y_m \geq x_m - M$. Altogether:

$$\bigwedge x_j \leq x_m \leq \bigwedge y_j + M + \delta$$

Now let $\delta \rightarrow 0$. The other inequality works the same way. \square

Next we define a special version of rough isometry, suiting the lattice structure of Lipschitz function spaces. The main new property will be an “approximate lattice homomorphism”. It exists in various versions, as Thomas Schick pointed out to us:

Proposition 13 *Let X, Y be (possibly infinite) metric spaces and $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ an ϵ -isometric embedding, $\epsilon \geq 0$. Then the following properties are equivalent:*

1. $f \leq g \Rightarrow \kappa f \leq \kappa g + \epsilon$ for all $f, g \in \text{Lip } Y$

2. $d_\infty((\kappa f) \vee (\kappa g), \kappa(f \vee g)) \leq \epsilon$ for all $f, g \in \text{Lip } Y$
3. For all $f_j \in \text{Lip } Y$, $j \in J$, $J \neq \emptyset$ some index set, holds:

$$d_\infty\left(\bigvee_{j \in J} \kappa f_j, \kappa \bigvee_{j \in J} f_j\right) \leq \epsilon \quad \text{and} \quad d_\infty\left(\bigwedge_{j \in J} \kappa f_j, \kappa \bigwedge_{j \in J} f_j\right) \leq \epsilon$$

Proof (3) \Rightarrow (2): $\#J = 2$.

(2) \Rightarrow (1): Assume there is some $x \in X$ such that $(\kappa f)(x) > (\kappa g)(x) + \epsilon$. From $f \leq g$ follows $f \vee g = g$, thus $d_\infty((\kappa f) \vee (\kappa g), \kappa g) \leq \epsilon$, in particular $(\kappa f)(x) \vee (\kappa g)(x) \leq (\kappa g)(x) + \epsilon$, contradiction.

(1) \Rightarrow (3): Obviously $f_k \leq \bigvee_{j \in J} f_j$ for all $k \in J$, hence $\kappa f_k \leq \kappa \bigvee_{j \in J} f_j + \epsilon$. We calculate the supremum over all $k \in J$: $\bigvee_j \kappa f_j \leq \kappa \bigvee_j f_j + \epsilon$. On the other hand, for every $\delta > 0$ there is some $k \in J$ with $d_\infty(f_k, \bigvee_j f_j) \leq \delta$. As κ is an ϵ -isometric embedding, this yields $d_\infty(\kappa f_k, \kappa \bigvee_j f_j) \leq \delta + \epsilon$, hence

$$\kappa \bigvee_{j \in J} f_j \leq \kappa f_k + \epsilon + \delta \leq \bigvee_{j \in J} \kappa f_j + \epsilon + \delta.$$

Let $\delta \rightarrow 0$. The other approximation works analogously. \square

We extend property (3) from Proposition 13 to allow $J = \emptyset$, and use it to define the notion of ml-isomorphisms:

Definition 14 Let X, Y be (possibly infinite) metric spaces. An ϵ -ml-homomorphism, $\epsilon \geq 0$ is an ϵ -isometric embedding $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$, with

$$d_\infty\left(\bigvee_{j \in J} \kappa f_j, \kappa \bigvee_{j \in J} f_j\right) \leq \epsilon \quad \text{and} \quad d_\infty\left(\bigwedge_{j \in J} \kappa f_j, \kappa \bigwedge_{j \in J} f_j\right) \leq \epsilon$$

for all $f_j \in \text{Lip } Y$, $j \in J$, J some index set.

An ϵ -ml-isomorphism is a pair of ϵ -ml-homomorphisms $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ and $\kappa' : \text{Lip } X \rightarrow \text{Lip } Y$, s.t. $\kappa \circ \kappa'$ and $\kappa' \circ \kappa$ are ϵ -near their corresponding identities. When we speak of an “ ϵ -ml-isomorphism κ ”, the corresponding κ' shall always be implied.

Proposition 15 For any ϵ -ml-isomorphism κ holds $d_\infty(\kappa(0), 0) \leq \epsilon$.

Proof As Andreas Thom pointed out, this follows directly from Definition 14 when $J = \emptyset$. There's also a 3ϵ -proof avoiding empty index sets: Let $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ be an ϵ -ml-isomorphism. We certainly know $0 \wedge \kappa'(0) = 0 \in \text{Lip } Y$, hence

$$d_\infty(\kappa(0), \kappa(0) \wedge \kappa\kappa'(0)) \leq 2\epsilon.$$

Now apply Proposition 12 to see

$$d_\infty(\kappa(0) \wedge \kappa\kappa'(0), \kappa(0) \wedge 0) \leq \epsilon$$

and use $\kappa(0) \wedge 0 = 0 \in \text{Lip } X$. \square

Proposition 16 *A δ -surjective ϵ -ml-homomorphism $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ induces a $(2\epsilon + 2\delta)$ -ml-isomorphism (κ, κ') .*

Proof For each $f \in \text{Lip } X$ choose an element $\kappa'(f)$, s.t. $d(\kappa\kappa'f, f) \leq \delta$. We show that the pair (κ, κ') defines a $(2\epsilon + 3\delta)$ -ml-isomorphism. The first inequality in Definition 14 is standard in coarse geometry:

$$\begin{aligned} d_\infty(f, g) &\leq d_\infty(\kappa\kappa'f, \kappa\kappa'g) + 2\delta \leq d_\infty(\kappa'f, \kappa'g) + (\epsilon + 2\delta) \\ d_\infty(f, g) &\geq d_\infty(\kappa\kappa'f, \kappa\kappa'g) - 2\delta \geq d_\infty(\kappa'f, \kappa'g) - (\epsilon + 2\delta) \end{aligned}$$

for all $f, g \in \text{Lip } X$. We now show that κ' fulfills the second and third inequality as well. Both can be handled the same way:

$$\begin{aligned} d_\infty\left(\bigwedge \kappa'f_j, \kappa' \bigwedge f_j\right) &\leq d_\infty\left(\kappa \bigwedge \kappa'f_j, \kappa\kappa' \bigwedge f_j\right) + \epsilon \\ &\leq d_\infty\left(\kappa \bigwedge \kappa'f_j, \bigwedge f_j\right) + \epsilon + \delta \\ &\leq d_\infty\left(\bigwedge \kappa\kappa'f_j, \bigwedge f_j\right) + 2\epsilon + \delta \\ &\leq \bigvee d_\infty(\kappa\kappa'f_j, f_j) + 2\epsilon + \delta \\ &\leq \delta + 2\epsilon + \delta \quad \forall f_j \in \text{Lip } X, j \in J \end{aligned}$$

Here we used (i) κ is ϵ -isometric embedding, (ii) $\kappa\kappa'$ is near identity, (iii) κ is ml-homomorphism, (iv) Proposition 12, (v) $\kappa\kappa'$ is near identity.

Finally we show that $\kappa'\kappa$ is $(\epsilon + \delta)$ -near identity:

$$d_\infty(\kappa'\kappa f, f) \leq d_\infty(\kappa\kappa'(\kappa f), (\kappa f)) + \epsilon \leq \delta + \epsilon \quad \forall f \in \text{Lip } X.$$

□

$\text{Lip } X$ is no algebra, like e.g. $C(X)$. Thus we can't give a basis of functions and reconstruct $\text{Lip } X$ by linear combinations. However, we can use the lattice structure to give another kind of "basis" for $\text{Lip } X$: Minimal Lipschitz functions with a given value at a single point.

Definition 17 *Let $x, y \in X$ and $r \in \mathbb{Z}$ be arbitrary. Define $\Lambda(x, r) \in \text{Lip } X$ by $\Lambda(x, r)(y) := (r - d(x, y)) \vee 0$.*

Note that this definition applies to $r = \infty$ or $d(x, y) = \infty$ as well: If $d(x, y) = \infty$, we have $\Lambda(x, r)(y) = 0$, and if $r = \infty$:

$$\Lambda(x, \infty)(y) = \begin{cases} \infty & : d(x, y) \neq \infty \\ 0 & : d(x, y) = \infty \end{cases}$$

Λ -functions with $r = \infty$ will be called *infinite*, else *finite*. Infinite Λ -functions are infinitely high characteristic functions for X 's components.

Proposition 18 *Let $x, y \in X$, $r, s \in Z$. Then holds:*

$$\begin{aligned} d_\infty(\Lambda(x, r), \Lambda(y, s)) &= \begin{cases} r \vee s & : d(x, y) \geq r \wedge s \\ |r - s| + d(x, y) & : d(x, y) \leq r \wedge s < \infty \\ 0 & : d(x, y) < r \wedge s = \infty \end{cases} \\ &\leq |r - s| + d(x, y) \end{aligned}$$

Proof Note that if $d(x, y) = r \wedge s$ the first and second case coincide, as $|r - s| = r \vee s - r \wedge s$. Assume without restriction $r \leq s$. Let

$$\begin{aligned} fz &:= |(0 \vee (r - d(x, z))) - (0 \vee (s - d(y, z)))| \quad \forall z \in X \\ d &:= d_\infty(\Lambda(x, r), \Lambda(y, s)) = \bigvee_{z \in X} f(z). \end{aligned}$$

Let's start with infinite cases. If $r = s = d(x, y) = \infty$, we get $d = \infty$ on both sides. If $r = s = \infty$, $d(x, y) \neq \infty$, we get $d = 0$. This is correct, as in this case the Λ -functions are equal. If $s = \infty$, $r \neq \infty$ we get $d = \infty$ again, for each variant of $d(x, y)$. If $r, s \neq \infty$ but $d(x, y) = \infty$, the two Λ -functions have different components as support, and thus d becomes the maximum of the differences, this is s .

Now we assume $r, s, d(x, y) \neq \infty$. First case: $r \leq d(x, y)$. Then we have

$$d \geq fy = |s - (0 \vee (r - d(x, y)))| = s$$

In addition, we have $\Lambda(x, r)(z) \in [0, r]$, $\Lambda(y, s)(z) \in [0, s]$, thus $fz \leq r \vee s = s$. Hence $d = s$. Second case: $d(x, y) \leq r$.

$$d \geq fy = |s - (0 \vee (r - d(x, y)))| = s - r + d(x, y)$$

And:

$$\begin{aligned} fz &= |r - d(x, z) - s + d(y, z)| \leq |r - s| + |d(y, z) - d(x, z)| \\ &\leq |r - s| + d(x, y) = s - r + d(x, y) \quad \forall z \in X \end{aligned}$$

□

Corollary 19 *For all $x, y \in X$ holds:*

$$d(x, y) = \lim_{r \rightarrow \infty, r \neq \infty} d_\infty(\Lambda(x, r), \Lambda(y, r))$$

Proof Follows directly from Proposition 18. □

This Corollary points us at an interesting aspect of Λ -functions: When we analyse the metric space $X_r := \{\Lambda(x, r) : x \in X\}$ with metric d_∞ for a fixed $r \in \mathbb{R}_{>0}$, we find it naturally isometric to (X, d_r) with the cut-off-metric $d_r(x, y) := r \wedge d(x, y)$ for all $x, y \in X$. Only in the limit $r \rightarrow \infty$, d_∞

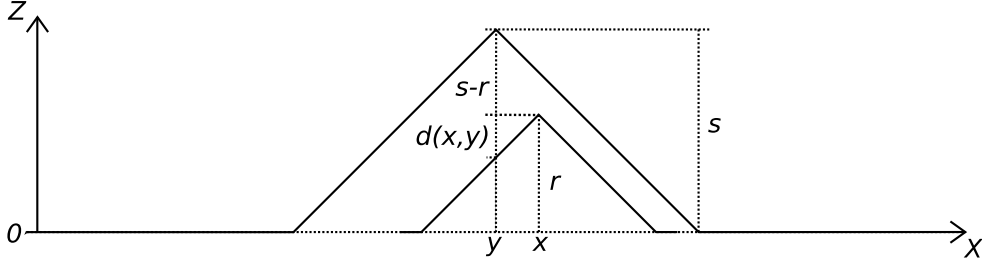


Figure 1: The d_∞ -distance between two Λ -functions is determined by their difference evaluated at the maximum point of the larger function, see Prop. 18.

will restore the full metric of X . Ironically, d_∞ obviously cuts away the coarse, large-scale information of X (in which we're primarily interested) and conserves the topological, small-scale informations. The large-scale information of X is still present, but more subtle to access.

Proposition 20 *For all $f \in \text{Lip } X$ holds: $f = \bigvee_{x \in X} \Lambda(x, f(x))$, where the latter is a pointwise maximum, not only supremum. If X is complete, the set $A := \{\Lambda(x, f(x)) : x \in X\} \cup \{0\}$ is (topologically) closed.*

Proof Let $g := \bigvee_{x \in X} \Lambda(x, f(x))$. Clearly, we have $\forall z \in X : fz \leq gz$, as $fz = \Lambda(z, fz)(z)$. We now observe that

$$fz \geq \Lambda(x, f(x))(z) \quad \forall x, z \in X.$$

For $d(x, z) \geq fx$, this is clear. For $d(x, z) \leq fx$, this follows from Lipschitz continuity ($fz \geq fx - d(x, z)$).

Furthermore, we notice that we deal with pointwise maxima: Each supremum of $\{\Lambda(x, f(x))(z)\}_{x \in X}$ is taken by $\Lambda(z, fz)(z) = fz$.

Let $\Lambda(x_j, fx_j)$ be any sequence converging to $g \in \text{Lip } X$, $x_j \in X$, $j \in \mathbb{N}$. First we notice

$$\begin{aligned} fx_j = d_\infty(0, \Lambda(x_j, fx_j)) &\geq d_\infty(0, g) - d_\infty(g, \Lambda(x_j, fx_j)) \\ \text{and } fx_j &\leq d_\infty(0, g) + d_\infty(g, \Lambda(x_j, fx_j)) \end{aligned}$$

hence $fx_j \rightarrow d_\infty(0, g)$. Assume $g \neq 0$ and finite. Then there is $x \in X$ with $gx > 0$ and fx_j must have a lower bound $R > 0$ for large enough j . By Cauchy criterion there is $N \in \mathbb{N}$ such that for all $j, k > N$ we have

$$d_\infty(\Lambda(x_j, fx_j), \Lambda(x_k, fx_k)) \leq \frac{1}{2} R < fx_j \wedge fx_k.$$

Due to Proposition 18 we conclude that for large enough j, k

$$d_\infty(\Lambda(x_j, fx_j), \Lambda(x_k, fx_k)) = |fx_j - fx_k| + d(x_j, x_k) \rightarrow 0.$$

Thus fx_j as well as x_j are Cauchy-sequences. As X and Z are metrically complete, we find $x' := \lim x_j$. As f is continuous, we have $fx' = \lim fx_j$,

and $\Lambda(x', fx') \in A$. Now we only have to show $g = \Lambda(x', fx')$. But this is clear, as for large enough j we have

$$d_\infty(\Lambda(x', fx'), \Lambda(x_j, fx_j)) \leq |fx' - fx_j| + d(x', x_j) \rightarrow 0 + 0.$$

Now assume g to be infinite (i.e. $\exists x : gx = \infty$). Then fx_j has to be infinite as well for large enough j (there is no non-trivial convergence to infinity in the chosen metric on Z) and Proposition 18 shows $d(x_j, x_k) < \infty$ for large enough j, k . Hence $\Lambda(x_j, fx_j) = \Lambda(x_k, fx_k) = g$. \square

We make some more use of the black magic of Proposition 12:

Proposition 21 *For all ϵ -isometries $\eta : X \rightarrow Y$ and $f \in \text{Lip } Y$ holds:*

$$d_\infty \left(\bigvee_{x \in X} \Lambda(x, f\eta x), \bigvee_{y \in Y} \Lambda(\eta' y, fy) \right) \leq \epsilon$$

Proof We observe that $d := d_\infty(\bigvee_{x \in X} \Lambda(x, f\eta x), \bigvee_{y \in Y} \Lambda(\eta' y, fy))$ can be rewritten to

$$d = d_\infty \left(\bigvee_{(x,y) \in J} \Lambda(x, f\eta x), \bigvee_{(x,y) \in J} \Lambda(\eta' y, fy) \right)$$

where $J := \{(x, y) \in X \times Y : y = \eta x \text{ or } x = \eta' y\}$: Each element of X (respectively Y) appears at least once in J , and multiple instances don't matter, as \bigvee is idempotent. Now Proposition 12 yields:

$$d \leq \bigvee_{(x,y) \in J} d_\infty(\Lambda(x, f\eta x), \Lambda(\eta' y, fy))$$

Let $(x, y) \in J$. Case 1: $y = \eta x$. Then

$$\begin{aligned} d_\infty(\Lambda(x, f\eta x), \Lambda(\eta' y, fy)) &= d_\infty(\Lambda(x, f\eta x), \Lambda(\eta' \eta x, f\eta x)) \\ &\leq d(x, \eta' \eta x) \leq \epsilon \end{aligned}$$

Case 2: $x = \eta' y$:

$$\begin{aligned} d_\infty(\Lambda(x, f\eta x), \Lambda(\eta' y, fy)) &= d_\infty(\Lambda(\eta' y, f\eta \eta' y), \Lambda(\eta' y, fy)) \\ &\leq |f\eta \eta' y - fy| \leq \epsilon \end{aligned}$$

\square

3 Inducing rough ml-isomorphisms

We make a first use of the notions of the preceding section. We proof that each ϵ -isometry $\eta : X \rightarrow Y$ lifts to an ϵ -isometry $\bar{\eta} : \text{Lip } Y \rightarrow \text{Lip } X$. Even better, $\bar{\eta}$ is an ϵ -ml-isomorphism, and is near $f \mapsto f \circ \eta$.

The next lemma is kind of a smoothening theorem. It states that the space of 1-Lipschitz functions over X is ϵ -dense in the space of $(1, \epsilon)$ -Lipschitz functions over X for all $\epsilon \geq 0$. A similar result for continuous functions is given by Petersen in [P], section 4.

Lemma 22 *Let $f \in \text{Lip}_{1,\epsilon}(X, Z)$. Define*

$$\bar{f} := \bigvee_{x \in X} \Lambda(x, fx)$$

Then f and \bar{f} are ϵ -near.

Proof We observe that $f(y)$ is never larger than $\bigvee_{x \in X} \Lambda(x, fx)(y)$ for all $y \in X$. So we have

$$d_\infty \left(f, \bigvee_{x \in X} \Lambda(x, fx) \right) = \bigvee_{x, y \in X} (\Lambda(x, fx)(y) - f(y))$$

and furthermore

$$\Lambda(x, fx)(y) - f(y) = \begin{cases} -f(y) & : d(x, y) \geq f(x) \\ f(x) - f(y) - d(x, y) & : d(x, y) \leq f(x) \end{cases}.$$

As $f(x) - f(y) - d(x, y) \leq \epsilon$ and $-f(y) \leq 0 \leq \epsilon$ we conclude the statement. (Note that each negative value is surpassed by at least one non-negative value, i.e. $-f(y)$ never occurs after taking the supremum.) \square

Proposition 23 *If $\eta : X \rightarrow Y$ is an ϵ -isometry, and $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ any mapping which is δ -near $f \mapsto f \circ \eta$, then κ is a $(2\epsilon + 2\delta)$ -ml-isomorphism.*

Proof (i) We show $|d_\infty(f, g) - d_\infty(\kappa f, \kappa g)| \leq 2\epsilon + 2\delta$ for all $f, g \in \text{Lip } X$. We have

$$|d_\infty(\kappa f, \kappa g) - d_\infty(f \circ \eta, g \circ \eta)| \leq 2\delta.$$

As next we notice $d_\infty(f \circ \eta, g \circ \eta) \leq d_\infty(f, g)$. Now let $x \in X$ be arbitrary, $y \in Y$ such that $d_Y(y, \eta x) \leq \epsilon$. Then $|f\eta x - fy| \leq \epsilon$ as f is 1-Lipschitz. Hence

$$\begin{aligned} |f\eta x - g\eta x| &\leq |fy - gy| + 2\epsilon \leq d_\infty(f, g) + 2\epsilon \\ \Rightarrow d_\infty(f \circ \eta, g \circ \eta) &\leq d_\infty(f, g) + 2\epsilon. \end{aligned}$$

(ii) For $J = \emptyset$ we observe that $d_\infty(\kappa(0), 0 \circ \eta) \leq \delta$ and $0 \circ \eta = 0$, as well as $d_\infty(\kappa(\infty), \infty \circ \eta) \leq \delta$ and $\infty \circ \eta = \infty$. Hence, assume $J \neq \emptyset$. We know

$$d_\infty\left(\kappa\left(\bigwedge f_j\right), \bigwedge(f_j \circ \eta)\right) = d_\infty\left(\kappa\left(\bigwedge f_j\right), \left(\bigwedge f_j\right) \circ \eta\right) \leq \delta$$

as the infimum is calculated pointwise. Hence, with Proposition 12:

$$\begin{aligned} d_\infty\left(\bigwedge(\kappa f_j), \kappa\left(\bigwedge f_j\right)\right) &\leq d_\infty\left(\bigwedge(\kappa f_j), \bigwedge(f_j \circ \eta)\right) + \delta \\ &\leq \bigvee d_\infty(\kappa f_j, f_j \circ \eta) + \delta \\ &\leq \epsilon + \delta. \end{aligned}$$

Same for supremum. \square

Theorem 24 (= Th. 1) *Given an ϵ -isometry $\eta : X \rightarrow Y$, $\bar{\eta}(f) := \overline{f \circ \eta}$ defines a 4ϵ -ml-isomorphism from $\text{Lip } Y$ to $\text{Lip } X$.*

Proof Let $f \in \text{Lip } Y$ be arbitrary. $f \circ \eta$ satisfies

$$d(f\eta x, f\eta y) \leq d(\eta x, \eta y) \leq d(x, y) + \epsilon.$$

Hence, $f \circ \eta$ and $\bar{\eta}(f)$ are ϵ -near (Lemma 22). However, $\bar{\eta}(f)$ is in $\text{Lip } X$, as it is a supremum of Lipschitz functions. Thus we can apply Proposition 23 to $\bar{\eta} : \text{Lip } Y \rightarrow \text{Lip } X$. Same holds for η' (Definition 7). It remains to show that $\bar{\eta} \circ \bar{\eta}'$ and $\bar{\eta}' \circ \bar{\eta}$ are near their respective identities.

We already saw that $\bar{\eta}(\bar{\eta}'(f))$ is ϵ -near $(\bar{\eta}'f) \circ \eta$. Similarly $\bar{\eta}'f$ is ϵ -near $f \circ \eta'$ and thus $(\bar{\eta}'f) \circ \eta$ is ϵ -near $f \circ \eta' \circ \eta$. Finally, $\eta' \circ \eta$ is ϵ -near identity, and as f is 1-Lipschitz, $f \circ \eta' \circ \eta$ is ϵ -near f , too. All this adds up to 3ϵ . Same for $\bar{\eta}' \circ \bar{\eta}$. \square

4 Inducing rough isometries

In this section, we show the reversal of Theorem 24: Given an ϵ -ml-isomorphism κ we construct a rough isometry η such that $\bar{\eta}$ is near κ .

Recall the definition of a join-irreducible:

$$f = g \vee h \Rightarrow f = g \vee f = h$$

It is interesting to see that the finite Λ -functions defined in Definition 17 satisfy a much more powerful version of join-irreducibility:

Lemma 25 *Let $p \in \text{Lip } Y$, Y complete. The following are equivalent:*

1. *p is a finite Λ -function, i.e. $\exists x \in Y, r \in Z \setminus \{\infty\} : p = \Lambda(x, r)$,*

2. $\forall (f_j)_{j \in J} \subseteq \text{Lip } Y, R \in Z :$

$$d_\infty \left(p, \bigvee_{j \in J} f_j \right) \leq R \Rightarrow \forall \delta > 0 \exists j \in J : d_\infty(p, f_j) \leq R + \delta$$

Proof In (2), the case $R = \infty$ is trivial. Hence, assume R to be finite.

(1) \Rightarrow (2): Let $(f_j)_{j \in J} \subseteq \text{Lip } Y$ and $R \geq 0$ be s.t. $d(p, \bigvee_{j \in J} f_j) \leq R$ holds. Choose $\delta > 0$ arbitrary and $p = \Lambda(y, s)$, $y \in Y$, $s \in Z \setminus \{\infty\}$. As

$$d \left(p(y), \bigvee f_j(y) \right) \leq R \Rightarrow p(y) - R - \delta < \bigvee f_j(y),$$

there has to be a $k \in J$ such that $p(y) - R - \delta < f_k(y)$, otherwise $p(y) - R - \delta$ would be a smaller upper bound for all f_j then $\bigvee f_j(y)$. From this, we see

$$f_k(x) \geq f_k(y) - d(x, y) > p(y) - d(x, y) - R - \delta.$$

Case 1: $p(y) \geq d(x, y)$. Then we have $p(x) = p(y) - d(x, y)$, and

$$f_k(x) > p(x) - R - \delta.$$

Case 2: $p(y) \leq d(x, y)$. Then $p(x) = 0$ and

$$f_k(x) \geq 0 > p(x) - R - \delta$$

holds trivially.

On the other hand, we have

$$f_k(x) \leq \bigvee_{j \in J} f_j(x) \leq p(x) + R < p(x) + R + \delta \quad \forall x \in Y$$

and thus $d_\infty(f_k, p) < R + \delta$.

(2) \Rightarrow (1): Choose $J = Y$, $f_y = \Lambda(y, p(y))$, $R = 0$, $\delta = 1/n$. This yields a sequence y_n of indices (= points in Y) such that $\Lambda(y_n, p(y_n)) \rightarrow p$. As $\{\Lambda(y, p(y)) : y \in Y\} \cup \{0\}$ is closed, we have either $p = \Lambda(y, p(y))$ for some $y \in Y$, or $p = 0 = \Lambda(y, 0)$ for any $y \in Y$. Now assume $p(y) = \infty$. Then

$$\begin{aligned} d_\infty \left(p, \bigvee_{r \in Z \setminus \{\infty\}} \Lambda(y, r) \right) &= 0 \\ \Rightarrow \exists r \in Z \setminus \{\infty\} : d_\infty(\Lambda(y, \infty), \Lambda(y, r)) &\leq 1 \end{aligned}$$

This is a contradiction to Proposition 18, hence p is a finite Λ -function. \square

Example 26 Lemma 25 does not hold for $\delta = 0$, just insert $\Lambda(y, 1) = \bigvee_{r \in (0,1)} \Lambda(y, r)$.

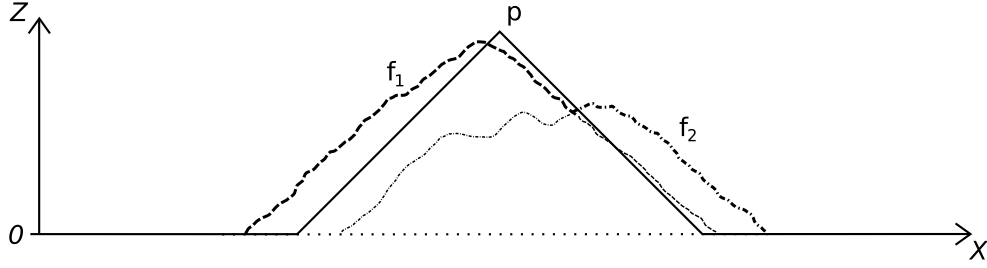


Figure 2: When approximating a Λ -function p by Lipschitz functions f_j , one of the functions (here f_1) must approximate the maximum point of p . This function may not decrease too fast (Lipschitz!), and may not increase too fast, as it is bounded from above by the approximation of p , hence it already approximates p on its own, see Lemma 25.

Recalling the short note after Corollary 19, the metric information of Y is encoded in the Λ -functions and the distances between them. However, these functions are at first sight just some arbitrary subset of $\text{Lip } Y$ and thus there's no hope for the metric space $(\text{Lip } Y, d_\infty)$ to hold the full information about Y 's metric. The preceding Lemma now explains to us, that the (finite) Λ -functions are not arbitrary at all – they have a specific, lattice theoretic property that distinguishes them from the remaining functions. Hence, in some sense the metric information of Y is now part of the combined metric and lattice structure of $\text{Lip } Y$.

Proposition 27 *Let Y be complete. Then the set $\{\Lambda(y, r) : y \in Y, r \in \mathbb{Z}\}$ of all Λ -functions in $\text{Lip } Y$ is topologically closed. In addition, the set of all finite Λ -functions is topologically closed.*

Proof Let $(f_j)_{j \in \mathbb{N}} = (\Lambda(x_j, r_j))_{j \in \mathbb{N}}$ be some sequence of Λ -elements in $\text{Lip } Y$ with limit f . If there is a subsequence $j(n)$ with $r_{j(n)} \rightarrow 0$ for $n \rightarrow \infty$, then this subsequence and hence (f_j) converges to $f = \Lambda(x, 0)$ for any $x \in Y$. So assume $\inf r_j$ is positive for large enough j . Due to Proposition 18 we have:

$$d(f_j, f_k) = \begin{cases} r_j \vee r_k & : d(x_j, x_k) \geq r_j \wedge r_k \\ |r_j - r_k| + d(x_j, x_k) & : d(x_j, x_k) \leq r_j \wedge r_k < \infty \\ 0 & : d(x_j, x_k) < r_j \wedge r_k = \infty \end{cases}$$

As the left side becomes arbitrarily small, whereas $r_j \vee r_k$ has a positive lower limit, only the second and third case may occur for $j, k \rightarrow \infty$. For large enough j, k , these cases don't mix anymore. The third case is trivial. From the second case we conclude $|r_j - r_k| \rightarrow 0$ and $d(x_j, x_k) \rightarrow 0$, and thus $r_j \rightarrow r$ and $x_j \rightarrow x$. Clearly, $f = \Lambda(x, r)$. In particular, r is finite in this case, which proves the second statement of the Proposition. \square

Lemma 28 *Let X, Y be complete, $\epsilon \geq 0$ and $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ an ϵ -ml-isomorphism. Then κ maps finite Λ -functions 6ϵ -near finite Λ -functions.*

Proof Let p be some finite Λ -function. Represent $\kappa(p)$ via Λ -functions $q_j, j \in J$. Let $\delta > 0$ be arbitrary. Then we have

$$\begin{aligned} d_\infty(\kappa(p), \bigvee q_j) &= 0 & | \text{ apply } \kappa' \\ \Rightarrow d_\infty(p, \bigvee \kappa'(q_j)) &\leq 3\epsilon \end{aligned}$$

Applying Lemma 25 to p , we know that there exists $k \in J$ such that

$$\begin{aligned} d_\infty(p, \kappa'(q_k)) &\leq 3\epsilon + \delta & | \text{ apply } \kappa \\ \Rightarrow d_\infty(\kappa(p), q_k) &\leq 5\epsilon + \delta. \end{aligned}$$

q_k must be a finite Λ -function, as

$$d_\infty(0, q_k) \leq d_\infty(0, \kappa(p)) + 5\epsilon + \delta \leq d_\infty(0, p) + 8\epsilon + \delta < \infty.$$

Case 1: $\epsilon > 0$. Choose $\delta = \epsilon$.

Case 2: $\epsilon = 0$. The preceding argument yields a sequence of finite Λ -functions converging to $\kappa(p)$. As of Proposition 27, $\kappa(p)$ must be a finite Λ -function as well. \square

The preceding Lemma is the critical point in our analysis: We can use Λ -functions as building blocks for Lipschitz functions, as Proposition 20 tells us. From Lemma 28 we now know that these building blocks (or, at least, the finite versions) behave sensible under ϵ -ml-isomorphisms κ , such that we only have to understand how they are mapped by κ to reconstruct all other Lipschitz functions. In particular, as they are strongly connected to the underlying spaces, they allow us to define mappings between them:

Lemma 29 *Let X, Y be complete, $\epsilon \geq 0$. Let $\kappa : \text{Lip } Y \rightarrow \text{Lip } X$ be an ϵ -ml-isomorphism, $\epsilon \geq 0$. Then there is a map $\eta : X \rightarrow Y$ such that*

$$d_\infty(\Lambda(\eta x, r), \kappa'(\Lambda(x, r))) \leq 59\epsilon$$

for all $x \in X, r \in Z$. For $r \in [38\epsilon, \infty)$, we may replace “59 ϵ ” by “43 ϵ ”.

Proof In the following proof, the first two cases will deal with $\epsilon > 0$ and finite r , the third with $\epsilon = 0$ and finite r and the fourth with $r = \infty$.

Case 1 and 2: For each $x \in X$, choose $\eta(x) \in Y$ and $s_x \in Z \setminus \{\infty\}$ such that $\Lambda(\eta x, s_x)$ is 6ϵ -near $\kappa' \Lambda(x, 22\epsilon)$ (use Lemma 28).

Case 1: $\epsilon > 0, r \in [38\epsilon, \infty)$. Let $\Lambda(x', r')$ be 6ϵ -near $\kappa' \Lambda(x, r)$. Then by Proposition 15 holds

$$\begin{aligned} d_\infty(0, \Lambda(x, r)) = r &\Rightarrow |d_\infty(0, \kappa' \Lambda(x, r)) - r| \leq 2\epsilon \\ &\Rightarrow |r' - r| \leq 8\epsilon. \end{aligned}$$

In the same way, we have

$$\begin{aligned} |d_\infty(0, \Lambda(\eta x, s_x)) - d_\infty(0, \kappa' \Lambda(x, 22\epsilon))| &\leq 6\epsilon \\ \Rightarrow |s_x - 22\epsilon| &\leq 8\epsilon. \end{aligned}$$

We now take a look at

$$\begin{aligned} d_\infty(\Lambda(x, r), \Lambda(x, 22\epsilon)) &= r - 22\epsilon \quad (\text{as } r \geq 22\epsilon) \\ \Rightarrow |d_\infty(\kappa' \Lambda(x, r), \kappa' \Lambda(x, 22\epsilon)) - (r - 22\epsilon)| &\leq \epsilon \\ \Rightarrow |d_\infty(\Lambda(x', r'), \Lambda(\eta x, s_x)) - (r - 22\epsilon)| &\leq 13\epsilon. \end{aligned}$$

Now we calculate $d := d_\infty(\Lambda(x', r'), \Lambda(\eta x, s_x))$ by hand. From Proposition 18, d could be $r' \vee s_x$ or $d(x', \eta x) + |r' - s_x|$. We know

$$s_x \leq 8\epsilon + 22\epsilon = 30\epsilon \leq r - 8\epsilon \leq r',$$

hence $r' \vee s_x = r'$. But, as $d \leq r - 22\epsilon + 13\epsilon = r - 9\epsilon$, but $r' \geq r - 8\epsilon$, d can't be r' (here we use $\epsilon > 0$). Remains

$$d = d(x', \eta x) + |r' - s_x| \quad \text{with} \quad |d - (r - 22\epsilon)| \leq 13\epsilon.$$

As shown above, $r' \geq s_x$, hence

$$\begin{aligned} d(x', \eta x) &\leq r - 22\epsilon + 13\epsilon - |r' - s_x| = r - 22\epsilon + 13\epsilon - r' + s_x \\ &\leq r - 22\epsilon + 13\epsilon - r + 8\epsilon + 22\epsilon + 8\epsilon = 29\epsilon. \end{aligned}$$

This, and $|r' - r| \leq 8\epsilon$, yield

$$\begin{aligned} d_\infty(\Lambda(\eta x, r), \Lambda(x', r')) &\leq d(x', \eta x) + |r - r'| \leq 37\epsilon \\ \Rightarrow d_\infty(\Lambda(\eta x, r), \kappa' \Lambda(x, r)) &\leq 43\epsilon. \end{aligned}$$

Case 2: $\epsilon > 0$, $r \in [0, 38\epsilon)$. Obviously,

$$\begin{aligned} d_\infty(\Lambda(\eta x, r), \kappa' \Lambda(x, r)) &\leq d_\infty(\Lambda(\eta x, r), \Lambda(\eta x, s_x)) \\ &\quad + d_\infty(\Lambda(\eta x, s_x), \kappa' \Lambda(x, 22\epsilon)) \\ &\quad + d_\infty(\kappa' \Lambda(x, 22\epsilon), \kappa' \Lambda(x, r)) \\ &\leq |r - s_x| + 6\epsilon + \epsilon + |r - 22\epsilon| \end{aligned}$$

As $r \in [0, 38\epsilon)$ and $s_x \in [14\epsilon, 30\epsilon]$ (see above), we receive $|r - s_x| \leq 30\epsilon$ and $|r - 22\epsilon| \leq 22\epsilon$. This adds up to 59ϵ .

Case 3: $\epsilon = 0$, $r \in [0, \infty)$. As of Lemma 28, for all $x \in X$ we can choose $\eta(x)$ such that $\kappa' \Lambda(x, 1) = \Lambda(\eta x, s_x)$ for some $s_x \in Z \setminus \{\infty\}$. From Proposition 15 we see $\kappa'(0) = 0$, hence $s_x = 1$. Now, let $r \in [0, \infty)$ be arbitrary. Let $x' \in Y$, $r' \in Z$ such that $\kappa' \Lambda(x, r) = \Lambda(x', r')$. Clearly, from the distance to 0 we again have $r' = r$. From

$$d_\infty(\Lambda(x, 1), \Lambda(x, r)) = |r - 1|$$

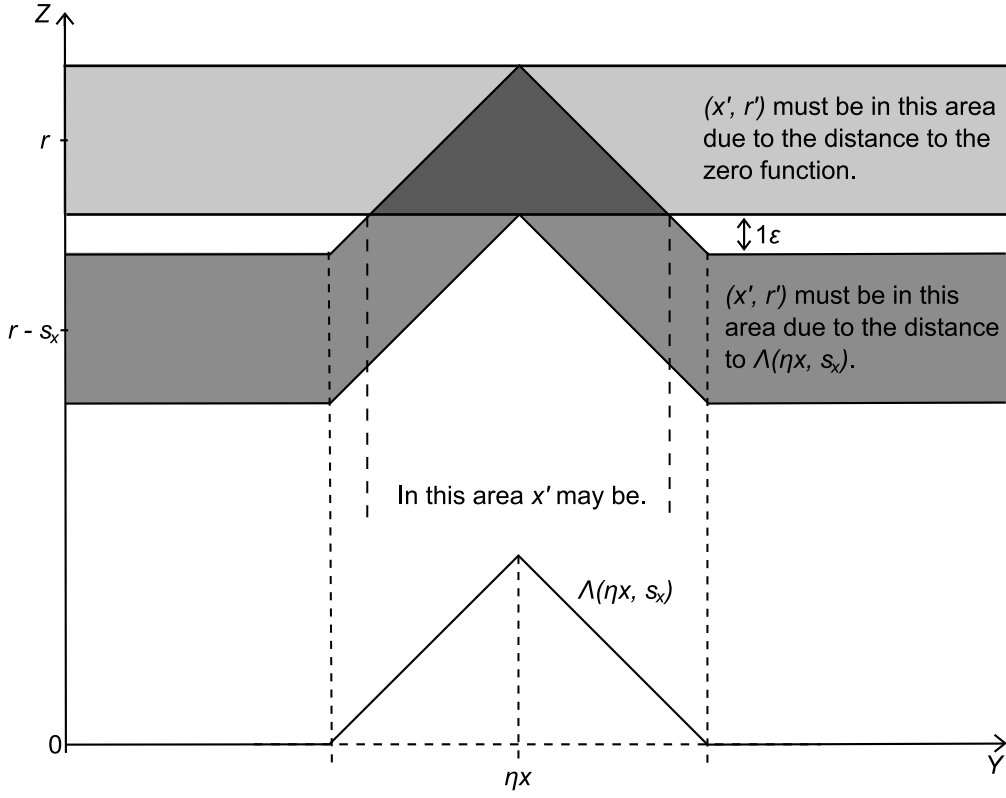


Figure 3: The function $\Lambda(x', r')$ in the proof of Lemma 29 is already determined up to nearness by its distance to two other functions: the zero function and $\Lambda(\eta x, s_x)$. This shows: A Λ -function $\Lambda(y, s)$ is not only mapped near another Λ -function $\Lambda(y', s')$, but y' only depends on y and s' only depends on s (modulo some multiples of ϵ).

we conclude

$$d_\infty(\Lambda(\eta x, 1), \Lambda(x', r)) = |r - 1|.$$

Due to Proposition 18 this can happen iff (a) $|r - 1| = 1 \geq r$ or (b) $|r - 1| = r \geq 1$ or (c) $d(x', \eta x) = 0$. Case (c) proves our statement, case (b) can't happen: $|r - 1| = r$ iff $r = \frac{1}{2}$, which contradicts $r \geq 1$. So, assume case (a). Then $r = 2$, which contradicts $r \leq 1$, or $r = 0$. But the case $r = 0$ is trivial, as we already saw from Proposition 15 that

$$\kappa' \Lambda(x, 0) = 0 = \Lambda(\eta x, 0).$$

Case 4: $r = \infty$. We know $\Lambda(x, \infty) = \bigvee_{s \in [38\epsilon, \infty)} \Lambda(x, s)$. Using our result for finite r , we conclude

$$d_\infty \left(\kappa' \bigvee_{s \in [38\epsilon, \infty)} \Lambda(x, s), \bigvee_{s \in [38\epsilon, \infty)} \Lambda(\eta x, s) \right) \leq 1\epsilon + 43\epsilon.$$

Apply $\bigvee_{s \in [38\epsilon, \infty)} \Lambda(\eta x, s) = \Lambda(\eta x, \infty)$ to see that $\kappa' \Lambda(x, \infty)$ is 44ϵ -near $\Lambda(\eta x, \infty)$. \square

Lemma 30 $\eta : X \rightarrow Y$ as defined in the proof of Lemma 29 is a 88ϵ -isometry.

Proof From Corollary 19 follows

$$d(\eta x, \eta y) = \lim_{r \rightarrow \infty, r \neq \infty} d_\infty(\Lambda(\eta x, r), \Lambda(\eta y, r)).$$

Applying Lemma 29 for large enough r yields:

$$|d_\infty(\Lambda(\eta x, r), \Lambda(\eta y, r)) - d_\infty(\kappa' \Lambda(x, r), \kappa' \Lambda(y, r))| \leq 2 \cdot 43\epsilon$$

And of course:

$$|d_\infty(\kappa' \Lambda(x, r), \kappa' \Lambda(y, r)) - d_\infty(\Lambda(x, r), \Lambda(y, r))| \leq \epsilon$$

Hence

$$|d(\eta x, \eta y) - d(x, y)| \leq 87\epsilon,$$

i.e. η is a rough isometric embedding. Just as η was constructed from κ' , we construct η' from κ . It remains to show that $\eta \circ \eta'$ and $\eta' \circ \eta$ are near identities. Again, we make use of Corollary 19:

$$\begin{aligned} |d(\eta \eta' x, x) - \lim_{r \rightarrow \infty, r \neq \infty} d_\infty(\Lambda(\eta \eta' x, r), \Lambda(x, r))| &= 0 \\ \Rightarrow |d(\eta \eta' x, x) - \lim d_\infty(\kappa' \kappa \Lambda(x, r), \Lambda(x, r))| &\leq 2 \cdot 43\epsilon + \epsilon \\ &\Rightarrow d(\eta \eta' x, x) \leq 88\epsilon \end{aligned}$$

Same for $\eta' \circ \eta$. \square

Theorem 31 (= Th. 2) Let X, Y be complete (possibly infinite) metric spaces and $\epsilon \geq 0$. For each ϵ -ml-isomorphism $\kappa : \text{Lip}(Y) \rightarrow \underline{\text{Lip}}(X)$ there is a 88ϵ -isometry $\eta : X \rightarrow Y$, such that κ is 61ϵ -near $\bar{\eta} : f \mapsto f \circ \eta$.

Proof Construct η as in Lemma 29. It's a 88ϵ -isometry due to Lemma 30. It remains to show that κ is near $\bar{\eta}$: Let $f \in \text{Lip } Y$ be arbitrary. Represent f via Λ -functions as in Proposition 20. Obviously,

$$d_\infty \left(\kappa \bigvee_{y \in Y} \Lambda(y, fy), \bigvee_{y \in Y} \Lambda(\eta' y, fy) \right) \leq 1\epsilon + 59\epsilon$$

due to Lemma 29. Apply Proposition 21. \square

5 Scaling limits

Definition 32 *Let the rough distance $d_R(X, Y)$ between two (possibly infinite) metric spaces X and Y be the infimum over all $\epsilon \geq 0$ such that X and Y are ϵ -isometric, or ∞ if there are none. If $d_R(X, Y) = 0$, the spaces X and X' will be called pseudo-isometric.*

The rough distance fulfills triangle-inequality, as concatenation of an ϵ - and a δ -isometry is an $\epsilon + \delta$ -isometry. It is closely related to the Gromov-Hausdorff-Distance for compact spaces, but may differ in a variable between $1/2$ and 2 (i.e., they are Lipschitz-equivalent, see e.g. [G2], Proposition 3.5).

Pseudo-isometry is a little bit less than isometry. However, they are equivalent if only compact spaces are compared (e.g. [P], [G2]), or if we deal with simple graphs, due to their integer metric. A nice article about scaling limits, Gromov-Hausdorff distances and quasi-isometries in the case of graphs and Cayley graphs is [Re].

Definition 33 *Let \mathbb{M} be the non-small groupoid of all pseudo-isometry-classes of metric spaces with ϵ -isometries as morphisms. d_R is a (possibly infinite) metric on \mathbb{M} in a natural way.*

Each of the components of \mathbb{M} can be endowed with a metric and topology, with the only drawback of being proper classes. This “topology” allows us to define the convergence of metric spaces to another metric space, up to pseudo-isometry. \mathbb{M} is complete in this “topology” (cf. [P], Proposition 6, the proof works in non-compact and non-separable cases as well).

Definition 34 *Let $\ell > 0$, and $s_\ell : \mathbb{M} \rightarrow \mathbb{M}$ given by*

$$s_\ell[(X, d)] := [(X, \ell \cdot d)]$$

which scales each metric space in \mathbb{M} by the factor ℓ ($\ell \cdot \infty := \infty$). This operation clearly is compatible with pseudo-isometry. Let $[X]$ be a space in \mathbb{M} . If the limit

$$s[X] := \lim_{\ell \rightarrow 0} s_\ell[X]$$

exists for all sequences $\ell \rightarrow 0$, then $s[X]$ (resp. all members of $s[X]$) is called the (strong) scaling limit of $[X]$.

We now want to apply Theorem 24.

Corollary 35 *Let X, Y be some (possibly infinite) metric spaces, such that Y is a strong scaling limit of X (Y is unique up to pseudo-isometry). Then there is a strong scaling limit of $\text{Lip } X$, and it is pseudo-isometric to $\text{Lip } Y$. (“The scaling limit of the Lipschitz space is the Lipschitz space of the scaling limit.”)*

Proof As $d_R(Y, s_\ell X) \rightarrow 0$ for $\ell \rightarrow 0$, there are ϵ_ℓ -isometries $\eta_\ell : s_\ell X \rightarrow Y$ with $\epsilon_\ell \rightarrow 0$. These induce $4\epsilon_\ell$ -ml-isomorphisms $\bar{\eta}_\ell : \text{Lip } Y \rightarrow \text{Lip } s_\ell X$, which are in particular $4\epsilon_\ell$ -isometries. Hence, $d_R(\text{Lip } Y, \text{Lip } s_\ell X) \rightarrow 0$. Proper rescaling of the associated Lipschitz functions further shows $s_\ell \text{Lip } X$ is naturally isometric to $\text{Lip } s_\ell X$, hence $s_\ell \text{Lip } X \rightarrow \text{Lip } Y$ up to pseudo-isometry. \square

Note that we can restrict to a set of \mathbb{M} when calculating a scaling limit. Thus, we can make use of Banach's fixed point theorem if d_R restricts to a true metric on this set.

6 Perspectives

6.1 Generalizations

There are several obvious ways to generalize the two main theorems: Changing the target space Z or the metric on $\text{Lip } X$ would break the main points of the proof, however single ideas might survive. The use of other types of functions is a similarly difficult question:

Example 36 Take $X = \{0\} \subseteq Y = \{0, 1\} \subseteq \mathbb{R}$ and η the inclusion, $\epsilon = 1$. The metric spaces of Z -valued continuous functions $C(X)$ and $C(Y)$ with sup-norm are isomorphic to Z and Z^2 respectively, which are not roughly isometric.

Another point is the inclusion of quasi-isometries. Although many ideas still work in the context of quasi-isometries, a function's Lipschitz constant is distorted in the process of Lemma 22. Hence there happens to be a “mixing” of the Lipschitz function spaces $\text{Lip}_K X$, which creates deep problems and at the same time great potential: If we find a workable solution to this problem, a new class of function spaces for groups would emerge, “quasi-Lipschitz functions”, so to speak.

Another very promising approach is to explore the rough isometries of Hajlasz-Sobolev spaces ([H], chapter 5). These are subsets of L^p function spaces, with a norm similar to the Sobolev norm. This norm contains a version of derivative which might compensate the obstruction we encounter with functions of arbitrary Lipschitz constant, at least for $p = \infty$.

6.2 Category Interpretation

Let X, Y be (possibly infinite) metric spaces. We define

$$d_{ml}(\text{Lip } X, \text{Lip } Y) := \inf\{\epsilon \geq 0 : \exists \kappa : \text{Lip } Y \rightarrow \text{Lip } X \text{ } \epsilon\text{-ml-isom.}\}$$

and $\text{Lip } \mathbb{M} := \{\text{Lip } X : X \text{ (poss.inf.) compl. metric space}\} / (d_{ml} = 0)$

$\text{Lip } \mathbb{M}$ is well-defined as $d_R(X, Y) = 0$ iff $d_{ml}(\text{Lip } X, \text{Lip } Y) = 0$ and because each pseudo-isometry-class contains a complete metric space. $\text{Lip } \mathbb{M}$ is a non-small groupoid with ml -isomorphisms as morphisms. In these terms, the mapping $\bar{\cdot} : \eta \mapsto \bar{\eta}$ is a Lipschitz equivalence between the metric categories \mathbb{M} and $\text{Lip } \mathbb{M}$, and a contravariant functor up to nearness of rough isometries.

6.3 Further Remarks

The proofs we presented here not only make use of the lattice structure of $\text{Lip } X$, but of a metric on it as well. In this sense, the comparison with Kaplanskys Theorem 4 is inconsistent. Indeed, already a simple scaling argument shows that we can't fully dispense with a structure besides the lattice to reconstruct all rough isometries. Thus, how much of the coarse geometry is really encoded in the lattice alone, and what else do we need to reconstruct rough or quasi-isometries? E.g., does the addition of the "Lipschitzized scaling"

$$\alpha_\ell : \text{Lip } X \rightarrow \text{Lip } X, \quad f \mapsto \bigvee_{x \in X} \Lambda(x, \ell \cdot fx)$$

for $\ell \geq 0$ as a structural component already suffice?

Finally, note the similarity of Definition 14 and the definition of Ulam's approximate group homomorphisms in [U], section VI.1; see [HR] for a survey on this topic. Indeed, we can state the question of stability of ml -homomorphisms and this directly corresponds to the rigidity of rough isometries through our main theorems.

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